ON THE COMPLETE INTEGRABILITY OF NONLINEAR DYNAMICAL SYSTEMS ON DISCRETE MANIFOLDS WITHIN THE GRADIENT-HOLONOMIC APPROACH

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Abstract. A gradient-holonomic approach for the Lax type integrability analysis of differential-discrete dynamical systems is described. The asymptotical solutions to the related Lax equation are studied, the related gradient identity subject to its relationship to a suitable Lax type spectral problem is analyzed in detail. The integrability of the discrete nonlinear Schrödinger, Ragnisco-Tu and Burgers-Riemann type dynamical systems is treated, in particular, their conservation laws, compatible Poissonian structures and discrete Lax type spectral problems are obtained within the gradient-holonomic approach.

1. Preliminary notions and definitions

Consider an infinite dimensional discrete manifold $M \subset l_2(\mathbb{Z};\mathbb{C}^m)$ for some integer $m \in \mathbb{Z}_+$ and a general nonlinear dynamical system on it in the form

$$dw/dt = K[w], \tag{1.1}$$

where $w \in M$ and $K : M \to T(M)$ is a Frechet smooth nonlinear local functional on M and $t \in \mathbb{R}$ is the evolution parameter. As examples of dynamical systems (1.1) on a discrete manifold $M \subset l_2(\mathbb{Z};\mathbb{C}^2)$ one can consider the well-known [5, 11] discrete nonlinear Schrödinger equation

$$\left.\begin{array}{c} du_n/dt = i(u_{n+1} - 2u_n + u_{n-1}) - iv_n u_n(u_{n+1} + u_{n-1}) \\ dv_n/dt = -i(v_{n+1} - 2v_n + v_{n-1}) + iv_n u_n(v_{n+1} + v_{n-1}) \end{array}\right\} := K_n[u,v], \tag{1.2}$$

the so called Ragnisco-Tu [26] equation

$$\left.\begin{array}{c} du_n/dt = u_{n+1} - u_n^2 v_n \\ dv_v/dt = -v_{n-1} + u_n v_n^2 \end{array}\right\} := K_n[u,v], \tag{1.3}$$

where we put $w := (u,v)^\intercal \in M$, and the inviscid Riemann-Burgers equation [20] on a discrete manifold $M \subset l_2(\mathbb{Z};\mathbb{R})$:

$$dw_n/dt = w_n(w_{n+1} - w_{n-1})/2 := K_n[w] \tag{1.4}$$

and its Riemann type [31, 29] generalizations, where $w \in M$, having applications [9] in diverse physics investigations.

For studying the integrability properties of differential-difference dynamical system (1.1) we will develop below a gradient-holonomic scheme before devised in [6, 15, 13, 7] for nonlinear dynamical systems defined on spatially one-dimensional functional manifolds and extended in [12] on the case of discrete manifolds.

Denote by $(\cdot,\cdot)$ the standard bi-linear form on the space $T^*(M) \times T(M)$ naturally induced by that existing in the Hilbert space $l_2(\mathbb{Z};\mathbb{C}^m)$. Having denoted by $\mathcal{D}(M)$ smooth functionals on M, for any functional $\gamma \in \mathcal{D}(M)$ one can define the gradient grad $\gamma[w] \in T^*(M)$ as follows:

$$\text{grad } \gamma[w] := \gamma^{\prime,*}[w] \cdot 1, \tag{1.5}$$

where the dash-sign "$'$" means the corresponding Frechet derivative and the star-sign " $*$ " means the conjugation naturally related with the bracket on $T^*(M) \times T(M)$.

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Definition 1.1. A linear smooth operator $\vartheta : T^*(M) \times T(M)$ is called *Poissonian* on the manifold M, if the bi-linear bracket

$$\{\cdot,\cdot\}_\vartheta := (\text{grad }(\cdot), \vartheta\text{grad }(\cdot)) \tag{1.6}$$

satisfies [1, 2, 8, 18, 6] the Jacobi identity on the space of functionals $\mathcal{D}(M)$.

This means, in particular, that bracket (1.5) satisfies the standard Jacobi identity on $\mathcal{D}(M)$.

Definition 1.2. A linear smooth operator $\vartheta : T^*(M) \times T(M)$ is called Nötherian [18, 6, 8] subject to the nonlinear dynamical system (1.1), if the following condition

$$L_K\vartheta = \vartheta^{'}K - \vartheta K^{\prime,*} - K^\prime\vartheta = 0 \tag{1.7}$$

holds identically on the manifold M, where we denoted by L_K the corresponding Lie-derivative [1, 2, 8, 6] along the vector field $K : M \to T(M)$.

Assume now that the mapping $\vartheta : T^*(M) \times T(M)$ is invertible, that is there exists the inverse mapping $\vartheta^{-1} := \Omega : T^*(M) \times T(M)$, and called symplectic. It then follows easily from (1.7) then the condition

$$L_K\Omega = \Omega^{'}K + \Omega K^\prime + K^{\prime,*}\Omega = 0 \tag{1.8}$$

hold identically on M. Having now assumed that the manifold $M \subset l_2(\mathbb{Z};\mathbb{C}^2)$ is endowed with a smooth Poissonian structure $\vartheta : T^*(M) \times T(M)$ one can define the Hamiltonian system

$$dw/dt := -\vartheta \text{ grad } H[w], \tag{1.9}$$

corresponding to a Hamiltonian function $H \in \mathcal{D}(M)$. As a simple corollary of definition (1.9) one obtains that the dynamical system

$$-\vartheta \text{ grad } H[w] := K[w] \tag{1.10}$$

satisfies the Nötherian conditions (1.7). Keeping in mind the study of the integrability problem [2, 4, 13, 8] subject to discrete dynamical system (1.1), we need to construct a priori given set of invariant with respect to it functions, called conservation laws, and commuting to each other with respect to the Poisson bracket (1.5). The following Lax criterion [3, 6, 13] proves to be very useful.

Lemma 1.3. *Any smooth solution $\varphi \in T^*(M)$ to the Lax equation*

$$L_K\ \varphi = d\varphi/dt + K^{\prime,*}\varphi = 0, \tag{1.11}$$

satisfying with respect to the bracket $(\cdot,\cdot)$ the symmetry condition

$$\varphi^{'} = \varphi^{'*},$$

is related to the conservation law

$$\gamma := \int_0^1 d\lambda(\varphi[w\lambda], w). \tag{1.12}$$

Proof. The expression (1.12) easily obtains from the well-known Volterra homology equalities:

$$\gamma = \int_0^1 \frac{d\gamma[w\lambda]}{d\lambda}d\lambda = \int_0^1 d\lambda(1, \gamma^\prime[w\lambda]\cdot w,) = \int_0^1 d\lambda(\gamma^{\prime,*}[w\lambda]\cdot 1, w) = \int_0^1 d\lambda(\text{grad }\gamma[w\lambda], w) \tag{1.13}$$

and

$$(\text{grad }\gamma[w])^{'} = (\text{grad }\gamma[w])^{\prime,*}, \tag{1.14}$$

holding identically on M. Whence one ensues that there exists a function $\gamma \in \mathcal{D}(M)$, such that

$$L_K\gamma = 0, \qquad \text{grad }\gamma[w] = \varphi[w] \tag{1.15}$$

for any $w \in M$. □

The Lax lemma naturally arises from the following generalized Nöther type lemma.

Lemma 1.4. *Let a smooth element $\psi \in T^*(M)$ satisfy the Nöther condition*

$$L_K\psi = d\psi/dt + K^{\prime,*}\psi = \text{grad } \mathcal{L}_\psi \tag{1.16}$$

for some smooth functional $\mathcal{L}_\psi \in \mathcal{D}(M)$. Then the following Hamiltonian representation

$$K = -\vartheta \text{ grad } H_\vartheta \tag{1.17}$$

holds, where

$$\vartheta := \psi^{'} - \psi^{'*} \tag{1.18}$$

and the Hamiltonian function

$$H_\vartheta = (\psi, K) - \mathcal{L}_\psi. \tag{1.19}$$

It is easy to see that Lemma 1.3 follows from Lemma 1.4, if the conditions $\psi^{'} = \psi^{'*}$ and $\mathcal{L}_\psi = 0$ are imposed on (1.16).

Assume now that equation (1.16) allows an additional not symmetric smooth solution $\phi \in T^*(M)$:

$$L_K\phi = d\phi/dt + K^{\prime,*}\phi = \text{grad } \mathcal{L}_\phi. \tag{1.20}$$

This means that our system (1.1) is bi-Hamiltonian:

$$-\vartheta \text{ grad } H_\vartheta = K = -\eta \text{ grad } H_\eta, \tag{1.21}$$

where, by definition,

$$\eta := \phi^{'} - \phi^{'*}, \quad H_\eta = (\phi, K) - \mathcal{L}_\phi. \tag{1.22}$$

Definition 1.5. One says that two Poissonian structures $\vartheta, \eta : T^*(M) \to T(M)$ on M are compatible [10, 18, 6, 8], if for any $\lambda, \mu \in \mathbb{R}$ the linear combination $\lambda\vartheta + \mu\eta : T^*(M) \to T(M)$ will be also Poissonian on M.

It is easy to derive that this condition is satisfied if, for instance, there exist the inverse operator $\vartheta^{-1} : T(M) \to T^*(M)$ and the expression $\eta(\vartheta^{-1}\eta) : T^*(M) \to T(M)$ is also Poissonian on M.

Concerning the integrability problem posed for the infinite-dimensional dynamical system (1.1) on the discrete manifold M it is, in general, necessary, but not enough [4, 6, 13], to prove the existence of an infinite hierarchy of commuting to each other with respect to the Poissonian structure (1.5) conservation laws.

Since in the case of the Lax type integrability almost of (1.1) there exist compatible Poissonian structures and related hierarchies of conservation laws, we will further constrain our analysis by devising an integrability algorithm under a priori assumption that a given nonlinear dynamical system (1.1) on the manifold M is Lax type integrable. This means that it possesses a related Lax type representation in the following, generally written form:

$$\Delta f_n := f_{n+1} = l_n[w;\lambda] f_n, \tag{1.23}$$

where $f := \{f_n \in \mathbb{C}^r : n \in \mathbb{Z}\} \subset l_2(\mathbb{Z}; \mathbb{C}^r)$ for some integer $r \in \mathbb{Z}_+$ and matrices $l_n[w;\lambda] \in End\mathbb{C}^r$, $n \in \mathbb{Z}$, in (1.23) are local matrix-valued functionals on M, depending on the "spectral" parameter $\lambda \in \mathbb{C}$, invariant with respect to our dynamical system (1.1).

Taking into account that the Lax representation (1.23) is 'local' with respect to the discrete variable $n \in \mathbb{Z}$, we will assume for convenience, that our manifold $M := M_{(N)} \subset l_\infty(\mathbb{Z}; \mathbb{C}^m)$ is periodic with respect to the discrete index $n \in \mathbb{Z}_N$, that is for any $n \in \mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ and $\lambda \in \mathbb{C}$

$$l_n[w;\lambda] = l_{n+N}[w;\lambda] \tag{1.24}$$

for some integer $N \in \mathbb{Z}_+$. In this case the smooth functionals on $M_{(N)}$ can be represented as

$$\gamma := \sum_{n \in \mathbb{Z}_N} \gamma_n[w] \tag{1.25}$$

for some local 'Frechet' smooth densities $\gamma_n : M_{(N)} \to \mathbb{C}$, $n \in \mathbb{Z}_N$.

2. The integrability analysis: gradient-holonomic scheme

Consider the representation (1.23) and define its fundamental solution $F_{m,n}(\lambda) \in Aut(\mathbb{C}^r)$, $m, n \in \mathbb{Z}_N$, satisfying the equation

$$F_{m+1,n}(\lambda) = l_m[w;\lambda]F_{m,n}(\lambda) \tag{2.1}$$

and the condition

$$F_{m,n}(\lambda)|_{m=n} = \mathbf{1} \tag{2.2}$$

for all $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}_N$. Then the matrix function

$$S_n(\lambda) := F_{n+N,n}(\lambda) \tag{2.3}$$

is called the *monodromy* matrix for the linear equation (1.24) and satisfies for all $n \in \mathbb{Z}_N$ the following Novikov-Lax type relationship:

$$S_{n+1}(\lambda)l_n = l_n S_n(\lambda). \tag{2.4}$$

It easy to compute that $S_n(\lambda) := \prod_{k=\overleftarrow{0,N-1}} l_{n+k}[w;\lambda]$ owing to the periodical condition (1.24). Construct now the following generating functional:

$$\bar{\gamma}(\lambda) := trS_n(\lambda) \tag{2.5}$$

and assume that there exists its asymptotical expansion

$$\bar{\gamma}(\lambda) \simeq \sum_{j\in\mathbb{Z}_+} \bar{\gamma}_j \lambda^{j_0-j} \tag{2.6}$$

as $\lambda \to \infty$ for some fixed $j_0 \in \mathbb{Z}_+$. Then, owing to the evident condition

$$D_n\overline{\gamma}(\lambda) = 0 \tag{2.7}$$

for all $n \in \mathbb{Z}_N$, where we put, by definition, the 'discrete' derivative

$$D_n := \Delta - 1, \tag{2.8}$$

we obtain that all functionals $\bar{\gamma}_j \in \mathcal{D}(M_{(N)})$, $j \in \mathbb{Z}_+$, are independent of the discrete index $n \in \mathbb{Z}_N$ and are simultaneously conservation laws for the dynamical system (1.1).

Assume additionally that the following natural condition holds: the gradient vector

$$\bar{\varphi}_n(\lambda) := \text{grad } \bar{\gamma}(\lambda)_n[w] = tr\ l^{\prime,*}(S_n(\lambda)l_n^{-1}), \tag{2.9}$$

which solves the Lax determining equation (1.11), satisfies, owing to (2.4), for all $\lambda \in \mathbb{C}$ the next gradient relationship:

$$z(\lambda)\vartheta\ \bar{\varphi}(\lambda) = \eta\ \bar{\varphi}(\lambda), \tag{2.10}$$

where $z : \mathbb{C} \to \mathbb{C}$ is some meromorphic mapping, ϑ and $\eta : T^*(M_{(N)}) \to T(M_{(N)})$ are compatible Poissonian on the manifold $M_{(N)}$ and Nötherian operators with respect to the dynamical system (1.1). As a corollary of the above condition one follows easily that the generating functional $\overline{\gamma}(\lambda) \in \mathcal{D}(M_{(N)})$ satisfies the commutation relationships

$$\{\bar{\gamma}(\lambda), \bar{\gamma}(\mu)\}_\vartheta = 0 = \{\bar{\gamma}(\lambda), \bar{\gamma}(\mu)\}_\eta \tag{2.11}$$

for all $\lambda, \mu \in \mathbb{C}$. Thereby, if to define on the manifold $M_{(N)}$ a generating dynamical system

$$dw/d\tau := -\vartheta \text{ grad } \bar{\gamma}(\lambda)[w] \tag{2.12}$$

as $\lambda \to \infty$, it follows easily from (2.11) that the hierarchy of functionals (2.6) is its conservation law.

Since the existence of an infinite hierarchy of commuting to each other conservation laws is characteristic concerning the Lax type integrability of the nonlinear dynamical system (1.1), this property can be effectively implemented into the scheme of our analysis. Namely, the following statement holds.

Proposition 2.1. *The Lax equation (1.11) allows the following asymptotical as $\lambda \to \infty$ periodical solution $\varphi(\lambda) \in T^*(M_{(N)})$:*

$$\varphi_n(\lambda) \simeq a_n(\lambda) \exp[\omega(t;\lambda)] \prod_{j=0}^{n} \sigma_j(\lambda), \tag{2.13}$$

where for all $n \in \mathbb{Z}$

$$\begin{aligned} a_n(\lambda) \quad &: \; = (1, a_{(1),n}[w;\lambda], a_{(2),n}[w;\lambda], ..., a_{(m-1),n}[w;\lambda])^{\tau}, \\ a_{(k),n}(\lambda) \quad &\simeq \sum_{s\in\mathbb{Z}_+} a_{(k),n}^{(s)}[w]\lambda^{-s+\tilde{a}}, \sigma_j(\lambda) \simeq \sum_{s\in\mathbb{Z}_+} a_j^{(s)}[w]\lambda^{-s+\tilde{\sigma}}, \end{aligned} \tag{2.14}$$

$k = \overline{1, m-1}$ and $\omega(t;\cdot) : \mathbb{C} \to \mathbb{C}$, $t \in \mathbb{R}$, is some dispersion function. Moreover the functional $\gamma(\lambda) := \sum_{n\in\mathbb{Z}_N} \ln(\lambda^{-\tilde{\sigma}}\sigma_n[w;\lambda]) \in \mathcal{D}(M_{(N)})$ is a generating function of conservation laws for the dynamical system (1.1).

Proof. Owing to Lemma (1.3) and relationship (2.9) functional (2.5) is a conservation law for our dynamical system (1.1). Based now on expression (2.3) and equation (1.23) one arrives at the solution representation (2.13) to the Lax equation (1.11). Now, making use of the periodicity of the manifold $M_{(N)}$, we obtain from the period translation of (2.13) that the functional

$$\gamma(\lambda) := \sum_{n\in\mathbb{Z}_N} \ln(\lambda^{-\tilde{\sigma}}\sigma_n[w;\lambda]) \simeq \sum_{j\in\mathbb{Z}_+} \gamma_j \lambda^{-j} \tag{2.15}$$

generates an infinite hierarchy of conservation laws to (1.1), so finishing the proof. □

Thus, if we start the Lax type integrability analysis of a priori given nonlinear dynamical system (1.1), it is necessary, as the first step, to study the asymptotical solutions (2.13) to the corresponding Lax equation (1.11) and construct a related hierarchy of conservation laws in the functional form (2.15), taking into account expansions (2.14).

Remark 2.2. It is easy to observe that, owing to the arbitrariness of the period $N \in \mathbb{Z}_+$ of the manifold $M_{(N)}$, all of the finite-sum expressions obtained above can be generalized to the corresponding infinite dimensional manifold $M \subset l_2(\mathbb{Z};\mathbb{C}^m)$, if the corresponding infinite series persist to be convergent.

Since our dynamical system (1.1) under the above conditions is a bi-Hamiltonian flow on the manifold $M_{(N)}$, as the next step of the related compatible Poissonian or symplectic structures, satisfying, respectively, either equality (1.7) or equality (1.8). Before doing this, we need to formulate the following lemma.

Lemma 2.3. *All functionals $\gamma_j \in \mathcal{D}(M_{(N)})$, entering the expansion (2.15), are commuting to each other with respect to both Poissonian structures $\vartheta, \eta : T^*(M_{(N)}) \to T(M_{(N)})$, satisfying the gradient relationship(2.16).*

Proof. Based on representations (2.13) and (2.9) one obtains that there holds the asymptotical as $\lambda \to \infty$ relationship

$$\ln \bar{\gamma}(\lambda) \simeq \gamma(\lambda). \tag{2.16}$$

Since the generative function $\bar{\gamma}(\lambda) \in \mathcal{D}(M_{(N)})$ satisfies the commutation relationships (2.11), the same also holds, owing to (2.16), for the generating function $\gamma(\lambda) \in \mathcal{D}(M_{(N)})$, finishing the proof. □

Proceed now to constructing the related to dynamical system (1.1) Poissonian structures $\vartheta, \eta : T^*(M_{(N)}) \to T(M_{(N)})$. Note here, that these Poissonian structures are Nötherian also for the whole hierarchy of dynamical systems

$$dw/dt_j := -\vartheta \text{ grad } \gamma_j[w], \tag{2.17}$$

where $t_j \in \mathbb{R}, j \in \mathbb{Z}_+$, are the corresponding evolution parameters, and which, owing to (2.11), commute to each other on the manifold $M_{(N)}$. The latter makes it possible to apply Lemma 1.4 to arbitrary one of the dynamical systems (2.17), if the related vector fields commuting with (1.1) are supposed to be found before.

To solve analytically equation (1.16) subject to an element $\varphi \in T^*(M_{(N)})$ one can, in the case of a polynomial dynamical system (1.1), make use of the well known asymptotical small parameter method [15, 6]. If applying this approach, it is necessary to take into account the following expansions at zero - element $\overline{w} = 0 \in M_{(N)}$ with respect to the small parameter $\mu \to 0$:

$$\begin{aligned} w \quad &: = \mu w^{(1)}, \ \varphi[w] = \varphi^{(0)} + \mu\varphi^{(1)}[w] + \mu^2\varphi^{(2)}[w] + ..., \\ d/dt &= d/dt_0 + \mu d/dt_1 + \mu^2 d/dt_2 + ..., \\ K[w] &= \mu K^{(1)}[w] + \mu^{(2)} K^{(2)}[w] + ..., \\ K'[w] &= K'_0 + \mu K'_1[w] + \mu^2 K'_2[w] + ..., \\ \text{grad } \mathcal{L}[w] &= \text{grad } \mathcal{L}^{(0)} + \mu \text{grad } \mathcal{L}^{(1)}[w] + \mu^2 \text{grad } \mathcal{L}^{(2)}[w] + ... \, . \end{aligned} \tag{2.18}$$

Having solved the corresponding set of linear nonuniform functional equations

$$\begin{aligned} d\varphi^{(0)}/dt_0 + K_0'^{*}\varphi^{(0)} &= \text{grad } \mathcal{L}^{(0)}, \\ d\varphi^{(1)}/dt_0 + K_0'^{*}\varphi^{(1)} &= \text{grad } \mathcal{L}^{(1)} - K_0'^{*}\varphi^{(0)}, \\ d\varphi^{(2)}/dt_0 + K_0'^{*}\varphi^{(2)} &= \text{grad } \mathcal{L}^{(2)} - K_1'^{*}\varphi^{(1)} - K_2'^{*}\varphi^{(0)} \end{aligned} \tag{2.19}$$

and so on, by means the standard Fourier transform applied to the suitable N-periodical functions, one can obtain the related Poissonian structure as the series

$$\vartheta^{-1} = \varphi^{(0),\prime} - \varphi^{(0),\prime *} + \mu(\varphi^{(1),\prime} - \varphi^{(1),\prime *}) + ... \tag{2.20}$$

and next to put in (2.20) at the end $\mu = 1$.

Another direct way of obtaining an Poissonian operator $\vartheta : T^*(M_{(N)}) \to T(M_{(N)})$ for (1.1) is to solve by means of the same asymptotical small parameter approach the Nötherian equation (1.7), having preliminary reduced it to the following set of linear nonuniform equations:

$$\begin{aligned} \frac{d}{dt_0}(\vartheta_0\varphi^{(0)}) &= K'_0(\vartheta_0\varphi^{(0)}), \\ \frac{d}{dt_0}(\vartheta_1\varphi^{(0)}) &= K'_0(\vartheta_1\varphi^{(0)}) + \vartheta_0 K_1^{\prime,*}\varphi^{(0)} + K'_1\vartheta_0\varphi^{(0)}, \\ \frac{d}{dt_0}(\vartheta_2\varphi^{(0)}) &= K'_0(\vartheta_2\varphi^{(0)}) - \varphi^{(0)\prime}K^1 + \vartheta_0 K_2^{\prime,*}\varphi^{(0)} + \\ &\quad +\vartheta_1 K_1^{\prime,*}\varphi^{(0)} + \vartheta_2 K_0^{\prime,*}\varphi^{(0)} + K'_1\vartheta_1\varphi^{(0)} + K'_2\vartheta_0\varphi^{(0)}. \end{aligned} \tag{2.21}$$

Based now on the analytical expressions for actions $\vartheta_j : \varphi^{(0)} \to \vartheta_j\varphi^{(0)}$, $j \in \mathbb{Z}_+$, one can easily retrieve them in operator form from the expansion

$$\vartheta = \vartheta_0 + \mu\vartheta_1 + \mu^2\vartheta_2 + ..., \tag{2.22}$$

if to put at the end of calculations $\mu = 1$. Similarly one can also construct the second Poissonian operator $\eta : T^*(M_{(N)}) \to T(M_{(N)})$ for the nonlinear dynamical system (1.1).

Resuming up all this analysis described above, we can formulate the following proposition.

Proposition 2.4. *Let a nonlinear dynamical system (1.1) on a discrete manifold $M_{(N)}$ allow both a nontrivial symmetric solution $\varphi \in T^*(M_{(N)})$ to the Lax equation (1.11) in the asymptotical as $\lambda \to \infty$ form (2.13), generating an infinite hierarchy of nontrivial functionally independent conservation laws (2.15), and compatible nonsymmetric solutions ψ and $\phi \in T^*(M_{(N)})$ to the Nöther equations (1.16 and (1.20), respectively. Then this dynamical system is a Lax type integrable bi-Hamiltonian flow on $M_{(N)}$ with respect to two compatible Poissonian structures $\vartheta, \eta : T^*(M_{(N)}) \to T(M_{(N)})$, whose adjoint Lax type representation*

$$d\Lambda/dt = [\Lambda, K^{\prime,*}], \tag{2.23}$$

where $\Lambda := \vartheta^{-1}\eta$ is the so-called recursion operator, can be transformed, owing to the gradient relationship (2.10), to the standard discrete Lax type form

$$dl_n/dt = [p_n(l), l_n] + (D_n p_n(l))l_n \tag{2.24}$$

for some matrix $p_n(l) \in End\mathbb{C}^r$ describing the related to (1.23) temporal evolution

$$df_n/dt = p_n(l)f_n \tag{2.25}$$

for $f \in l_\infty(\mathbb{Z}; \mathbb{C}^r)$.

Remark 2.5. Based on the property that all Hamiltonian flows (2.17) commute to each other and to dynamical system (1.1) and using the fact that they possess the same Poissonian and compatible (ϑ,η)-pair, the analytical algorithm described above can be equally applied to any other flow, commuting with (1.1).

Concerning the discrete linear Lax type problem (1.23), it can be constructed by means of the gradient-holonomic algorithm, devised in [6, 7, 13] for studying the integrability of nonlinear dynamical systems on functional manifolds. Namely, making use of the preliminary found analytical expressions for the related compatible Poissonian structures $\vartheta,\eta : T^*(M_{(N)}) \to T(M_{(N)})$ on the manifold $M_{(N)}$ and using the fact that the recursion operator $\Lambda := \vartheta^{-1}\eta : T^*(M_{(N)}) \to T^*(M_{(N)})$ satisfies the dual Lax type commutator equality (2.23), one can retrieve the standard Lax type representation for it by means of suitably derived algebraic relationships. As a corollary of Proposition 2.4 one can also claim that the existence of a nontrivial asymptotical as $\lambda \to \infty$ solution to the Lax equation (1.11) can serve as an effective Lax type integrability criterion for a given nonlinear dynamical system (1.1) on the manifold $M_{(N)}$.

3. The Bogoyavlensky-Novikov finite-dimensional reduction

Assume that our dynamical system (1.1) on the periodic manifold $M_{(N)}$ is Lax type integrable and possesses two compatible Poissonian structures $\vartheta,\eta : T^*(M_{(N)}) \to T(M_{(N)})$.Thus, we have the nonlinear finite-dimensional dynamical system

$$dw_n/dt := K_n[w] = -\vartheta \text{ grad } H_n[w] \tag{3.1}$$

for indices $n \in \mathbb{Z}_N$ owing to its N-periodicity. The finite dimensional dynamical system (3.1) can be equivalently considered as that on the finite-dimensional space $M_{(N)} \simeq (\mathbb{C}^m)^N$ parameterized by any integer index $n \in \mathbb{Z}_N$, and whose Liouville integrability is of the next our analysis. To proceed with studying the flow (3.1) on the manifold $M_{(N)}$, we will make use of the Bogoyavlensky-Novikov [16, 4] reduction scheme [4, 12, 6, 8].

Let $\Lambda(M_{(N)}) := \underset{j\in\mathbb{Z}_+}{\oplus} \Lambda^j(M_{(N)})$ be the standard finitely generated Grassmann algebra [2, 6, 13] of differential forms on the manifold $M_{(N)}$. Then the following differential complex

$$\Lambda^0(M_{(N)}) \xrightarrow{d} \Lambda^1(M_{(N)}) \xrightarrow{d} ... \xrightarrow{d} \Lambda^j(M_{(N)}) \xrightarrow{d} \Lambda^{j+1}(M_{(N)}) \xrightarrow{d} ...., \tag{3.2}$$

where $d : \Lambda(M_{(N)}) \to \Lambda(M_{(N)})$ is the external differentiation, is finite and exact. Since the discrete 'derivative' $D_n := \Delta - 1$ commutes with the differentiation $d : \Lambda(M_{(N)}) \to \Lambda(M_{(N)})$, $[D_n, d] = 0$ for all $n \in \mathbb{Z}_N$, and for any element $a \in \Lambda^0(M_{(N)})$

$$\text{grad}(\sum_{n\in\mathbb{Z}_N} D_n a_n[w]) = 0, \tag{3.3}$$

one can formulate the following Gelfand-Dikiy type [17] lemma.

Lemma 3.1. *Let $\mathcal{L}[w] \in \Lambda^0(M_{(N)})$ be a Frechet smooth local Lagrangian functional on the manifold $M_{(N)}$. Then there exists a differential 1-form $\alpha^{(1)} \in \Lambda^1(M_{(N)})$, such that the equality*

$$d\mathcal{L}_n[w] =< \text{grad } \mathcal{L}_n[w], dw_n > + D_n\alpha_n^{(1)}[w] \tag{3.4}$$

holds for all $n \in \mathbb{Z}_N$.

Proof. One can easily observe that

$$\begin{aligned} d\mathcal{L}_n[w] &= \sum_{j=0}^{N-1} < \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}}, dw_{n+j} >= \sum_{j=0}^{N-1} < \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}}, \Delta^j dw_n >= \\ &= < \sum_{j=0}^{N-1} \Delta^{-j} \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}}, dw_n > + D_n \left(\sum_{j=0}^{N-1} < p_j, dw_{n+j} > \right), \end{aligned} \tag{3.5}$$

where, by definition,

$$p_k := \sum_{j=0}^{N-1} \Delta^{-j} \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j+k+1}} \tag{3.6}$$

for $k=\overline{0,N-1}$. Having denoted the expression

$$\text{grad } \mathcal{L}_n[w] := \sum_{j=0}^{N-1} \Delta^{-j} \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}}, \tag{3.7}$$

one obtains the result (3.4), where

$$\alpha_n^{(1)}[w] := \sum_{j=0}^{N-1} < p_j, dw_{n+j} > \tag{3.8}$$

is the corresponding differential 1-form on the manifold $M_{(N)}$. □

Applying now the d-differentiation to expression (3.4) we obtain that

$$-D_n \omega_n^{(2)}[w] =< d \text{ grad } \mathcal{L}_n[w], \wedge dw_n > \tag{3.9}$$

for any $n \in \mathbb{Z}$, where the 2-form

$$\omega^{(2)}[w] := d\alpha^{(1)}[w] \tag{3.10}$$

is nondegenerate on $M_{(N)}$, if the Hessian $\partial_n^2 \mathcal{L}[w]/\partial^2 w_n$ is also nondegenerate.

Assume now that the submanifold

$$\bar{M}_{(N)} := \left\{ \text{grad } \mathcal{L}_n^{(\bar{N})}[w] = 0; \ w \in M_{(N)} \right\}, \tag{3.11}$$

where, by definition, the Lagrangian functional

$$\mathcal{L}^{(\bar{N})} := -\gamma_{\bar{N}} + \sum_{j=0}^{\bar{N}-1} c_j \gamma_j, \tag{3.12}$$

with $\gamma_j \in \mathcal{D}(M)$, $j=\overline{0,\bar{N}-1}$, for some $\bar{N} \in \mathbb{Z}_+$, being the suitable nontrivial conservation laws for the dynamical system (1.1), constructed before, and $c_j \in \mathbb{C}$, $j=\overline{0,\bar{N}-1}$, being some arbitrary but fixed constants. As a result of (3.11) and (3.9) we obtain that closed 2-form $\omega^{(2)} \in \Lambda^2(\bar{M}_{(N)})$ is invariant with respect to the index $n \in \mathbb{Z}_N$ on the manifold $\bar{M}_{(N)}$. Moreover, the submanifold (3.11) is also invariant both with respect to the index $n \in \mathbb{Z}_N$ and the evolution parameter $t \in \mathbb{R}$. Really, for any $n \in \mathbb{Z}_N$ the Lie derivative

$$L_K \text{grad } \mathcal{L}^{(\bar{N})} = (\text{grad } \mathcal{L}^{(\bar{N})})' K + K'^{,*} (\text{grad } \mathcal{L}^{(\bar{N})}) = 0, \tag{3.13}$$

since the functional $\mathcal{L}^{(\bar{N})} \in \mathcal{D}(M_{(N)})$ is a sum of conservation laws for the dynamical system (1.1), whose gradients satisfies the Lax condition (1.11). Moreover, it is easy to see that if the Lie derivative L_K grad $\mathcal{L}_n^{(\bar{N})}[w]=0, n \in \mathbb{Z}_N$, at $t=0$, then grad $\mathcal{L}_n^{(\bar{N})}[w]=0$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}_N$. Thus, the Bogoyavlensky-Novikov reduction of the dynamical system (1.1) upon the invariant submanifold $\bar{M}_{(N)}$ is defined completely invariantly.

Now a question arises: how are related the dynamical system (1.1), naturally constrained to live on the submanifold $M_{(N)}$, and the dynamical system (1.1), reduced on the finite dimensional submanifold $\bar{M}_{(N)} \subset M_{(N)}$. To analyze this reduction we will consider the following equality:

$$< \text{grad } \mathcal{L}_n^{(\bar{N})}[w], K_n[w] >= -D_n h_n^{(t)}[w], \tag{3.14}$$

for some local functional $h^{(t)}[w] \in \Lambda^0(M)$, following from the conditions (3.3) and (1.11):

$$\begin{aligned} \text{(3.15)}\ \text{grad} \ &< \text{grad } \mathcal{L}_n^{(\bar{N})}[w], K_n[w] >= (\text{grad } \mathcal{L}_n^{(\bar{N})}[w])'^{,*} K_n[w] + K_n'^{,*}[w] \text{grad } \mathcal{L}_n^{(\bar{N})}[w] = \\ &= (\text{grad } \mathcal{L}_n^{(\bar{N})}[w])' K_n[w] + K_n'^{,*}[w] \text{grad } \mathcal{L}_n^{(\bar{N})}[w] = L_K \text{grad } \mathcal{L}_n^{(\bar{N})}[w] = 0, \end{aligned}$$

giving rise to (3.14). Since on the submanifold $\bar{M}_{(N)}$ the gradient grad $\mathcal{L}_n^{(\bar{N})}[w]=0$ for all $n \in \mathbb{Z}_N$, we obtain from (3.14) that the local functional $h^{(t)}[w] \in \Lambda^0(\bar{M}_{(N)})$ does not depend on index $n \in \mathbb{Z}_N$.

The properties of the manifold $\bar{M}_{(N)}$, described above, make it possible to consider it as a symplectic manifold endowed with the symplectic structure $\omega^{(2)} \in \Lambda^2(\bar{M}_{(N)})$, given by expressions (3.8) and (3.10). From this point of view we can proceed to studying the integrability properties of the dynamical system (1.1) reduced upon the invariant finite-dimensional manifold $\bar{M}_{(N)} \subset M_{(N)}$.

First, we observe that the vector field d/dt on $\bar{M}_{(N)}$ is canonically Hamiltonian [1, 2, 4] with respect to the symplectic structure $\omega^{(2)} \in \Lambda^2(\bar{M}_{(N)})$:

$$- i_{\frac{d}{dt}}\omega^{(2)}(w,p) = dh^{(t)}(w,p), \tag{3.16}$$

where $h^{(t)}(w,p) := h^{(t)}(w), \omega^{(2)}(w,p) := \omega^{(2)}[w]$ and $(w,p)^{\intercal} \in \bar{M}_{(N)}$ are canonical variables induced on the manifold $\bar{M}_{(N)}$ by the Liouville 1-form (3.8). Really, from expression (3.14) one obtains that

$$di_{\frac{d}{dt}} < \text{grad } \mathcal{L}_n^{(\bar{N})}[w], dw_n >= -D_n dh_n^{(t)}[w],$$

which, being supplemented with the identity (3.9) in the form

$$i_{\frac{d}{dt}} d < \text{grad } \mathcal{L}_n^{(\bar{N})}[w], dw_n >= -D_n i_{\frac{d}{dt}} \omega_n^{(2)}[w],$$

entails the following:

$$\frac{d}{dt} < \text{grad } \mathcal{L}_n^{(\bar{N})}[w], dw_n >= -D_n(dh_n^{(t)}[w] + i_{\frac{d}{dt}}\omega_n^{(2)}[w]), \tag{3.17}$$

Since grad $\mathcal{L}^{(\bar{N})}[w] = 0 = L_K$ grad $\mathcal{L}[w]$ identically on $\bar{M}_{(N)}$, from (3.17) one obtains the result (3.16).

The same one can claim subject to any of Hamiltonian systems (2.17), commuting with (1.1) on the manifold M. Moreover, owing to the functional independence of invariants $\gamma_j \in \mathcal{D}(M_{(N)})$, $j = \overline{0, \bar{N}-1}$, entering the Lagrangian functional (3.12), we can construct the set of functionally independent functions $h^{(j)} \in \mathcal{D}(\bar{M}_{(N)})$, $j = \overline{0, \bar{N}-1}$, as follows:

$$< \text{grad } \mathcal{L}_n^{(\bar{N})}[w], \vartheta_n \text{ grad } \gamma_{j,n}[w] >= D_n h_n^{(j)}[w], \tag{3.18}$$

It is easy to check that these functions $h^{(j)} \in \mathcal{D}(\bar{M}_{(N)}), j = \overline{0, \bar{N}-1}$, are invariant with respect to indices $n \in \mathbb{Z}_N$ and commuting both to each other and to the Hamiltonian function $h^{(t)} \in \mathcal{D}(\bar{M}_{(N)})$ with respect to the symplectic structure $\omega^{(2)} \in \Lambda^2(\bar{M}_{(N)})$. Thus, if the dimension $\dim \bar{M}_{(N)} = 2\bar{N}$, the discrete dynamical system (1.1) reduced upon the finite-dimensional submanifold $\bar{M}_{(N)} \subset M_{(N)}$ will be Liouville integrable. If the set of conservation laws $\gamma_j \in \mathcal{D}(M_{(N)})$, $j = \overline{0, \bar{N}-1}$, proves to be functionally dependent on $M_{(N)}$, the described scheme should be modified by means of using the Dirac reduction technique [1, 8, 6] for regular finding the symplectic structure $\bar{\omega}^{(2)}[w] \in \Lambda^2(\bar{M}_{(N)})$ on invariant nonsingular submanifolds.

4. Example 1: the differential-difference nonlinear Schrödinger dynamical system and its integrability

The mentioned before discrete nonlinear Schrödinger dynamical system (1.2) is defined on the periodic manifold $M_{(N)} \subset l_\infty(\mathbb{Z};\mathbb{C}^2)$. Its Lax type integrability was stated in [5, 11, 14] making use of the simplest discretization of the standard Zakharov-Shabat spectral problem for the well-known nonlinear Schrödinger equation. In this Section we will demonstrate the application of the gradient-holonomic integrability analysis, described above, to this discrete nonlinear Schrödinger dynamical system (1.2). First, we will show the existence of an infinite hierarchy of functionally independent conservation laws, having solved the determining Lax equation (1.11) in the asymptotical form (2.13). The following lemma holds.

Lemma 4.1. *The functional expression*

$$\varphi_n := \begin{pmatrix} 1 \\ a_n(\lambda) \end{pmatrix} \exp[it(2-\lambda-\lambda^{-1})] \prod_{j=1}^{n} \sigma_j(\lambda), \tag{4.1}$$

is an asymptotical, as $\lambda \to \infty$, *solution to the determining Lax equation*

$$d\varphi_n/dt + K_n^{\prime,*}\varphi_n = 0 \tag{4.2}$$

for all $n \in \mathbb{Z}_N$ *with the operator* $K^{\prime,*} : T^*(M_{(N)}) \to T^*(M_{(N)})$ *of the form:*

$$K_n^{\prime,*} = \begin{pmatrix} \begin{matrix} i\Delta^{-1}D_n^2 - iv_n(u_{n+1}+u_{n-1}) - \\ -i(\Delta+\Delta^{-1})\cdot v_n u_n \end{matrix} & iv_n(v_{n+1}+v_{n-1}) \\ -iu_n(u_{n+1}+u_{n-1}) & \begin{matrix} -i\Delta^{-1}D_n^2 + iu_n(v_{n+1}+v_{n-1}) + \\ +i(\Delta+\Delta^{-1})\cdot v_n u_n \end{matrix} \end{pmatrix}, \tag{4.3}$$

where, by definition,

$$
\begin{aligned}
\sigma_n(\lambda) &\simeq \frac{\lambda}{h_n[u,v]}(1-\sum_{s\in\mathbb{Z}_+}\sigma_{n+}^{(s)}[u,v]\lambda^{-s-1}), \\
a_n(\lambda) &\simeq \sum_{s\in\mathbb{Z}_+} a_n^{(s)}[u,v]\lambda^{-s}
\end{aligned}
\tag{4.4}
$$

are the corresponding asymptotical expansions.

Proof. To prove this Lemma it is enough to find the corresponding coefficients of the asymptotical expansions (4.4). To do this we will consider the following two equations easily obtained from (4.2), (4.3) and (4.1) :

$$
\begin{aligned}
&D_n^{-1}\tfrac{d}{dt}[-\ln h_n+\ln(1-\textstyle\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})]+ \\
&+i\lambda[h_{n+1}^{-1}(1-v_nu_n)(1-\textstyle\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})-1]+ \\
&+\tfrac{i}{\lambda}\left[(1-v_{n-1}u_{n-1})h_n(1-\textstyle\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})^{-1}-1\right]- \\
&-iv_n(u_{n+1}+u_{n-1})+iv_n(v_{n+1}+v_{n-1})\textstyle\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s}
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
&(\textstyle\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s})D_n^{-1}\tfrac{d}{dt}[-\ln h_n+\ln(1-\textstyle\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})]+4i(\textstyle\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s})+ \\
&+\left[i\lambda h_{n+1}(v_{n+1}u_{n+1}-1)(\textstyle\sum_{s\in\mathbb{Z}_+}a_{n+1}^{(s)}\lambda^{-s})(\textstyle\sum_{s\in\mathbb{Z}_+}a_{n+1}^{(s)}\lambda^{-s})-\textstyle\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s}\right]+ \\
&+\tfrac{i}{\lambda}\left[(v_{n-1}u_{n-1}-1)(\textstyle\sum_{s\in\mathbb{Z}_+}a_{n+1}^{(s)}\lambda^{-s})h_n(1-\textstyle\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})^{-1}-\textstyle\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s}\right]+ \\
&+\tfrac{d}{dt}\textstyle\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s}-iu_n(u_{n+1}+u_{n-1})+iu_n(v_{n+1}+v_{n-1})\textstyle\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s}.
\end{aligned}
\tag{4.6}
$$

Having equated the coefficients of (4.5) at the same degrees of the parameter $\lambda\in\mathbb{C}$, we obtain step-by-step the functional expressions for $h_n,\sigma_n^{(s)}$ and $a_n^{(s)}$

$$
\begin{aligned}
h_n &= (1-v_nu_n), a_n^{(0)}=0, a_n^{(1)}=\beta, \\
\sigma_n^{(0)} &= v_{n-1}u_{n-1}+v_{n-1}u_{n-2}(u_n+u_{n-2})-iD_n^2(\ln h_{n-1})_t, \\
\sigma_n^{(1)} &= i\frac{d}{dt}\sigma_{n-1}^{(0)}+(h_{n-1}h_{n-2}-1)+a_{n-1}^{(1)}v_{n-1}(u_n+u_{n-2}), \\
a_n^{(2)} &= -3a_{n-1}^{(1)}+i\frac{d}{dt}\sigma_{n-1}^{(1)}-ia_{n-1}^{(1)}D_n^{-1}(\ln h_{n-1})_t+ \\
&\quad +a_n^{(1)}\sigma_n^{(0)}-u_{n-1}(v_n+v_{n-2})a_{n-1}^{(1)}, \\
dh_n/dt &= iD_n(v_{n-1}u_n-v_nu_{n-1}),...,
\end{aligned}
\tag{4.7}
$$

for all $n\in\mathbb{Z}, s\in\mathbb{Z}$, or

$$
\begin{aligned}
\sigma_n^{(0)} &= v_{n-1}u_{n-1}+v_{n-1}u_{n-2}(u_n+u_{n-2})-iD_n^2(\ln h_{n-1})_t, \\
\sigma_n^{(1)} &= i\frac{d}{dt}\sigma_{n-1}^{(0)}+(1-v_{n-1}u_{n-1})(1-v_{n-2}u_{n-2})+\beta v_{n-1}(u_n+u_{n-2}),...,
\end{aligned}
\tag{4.8}
$$

and so on. Thus, having stated that the corresponding iterative equations are solvable for all $s\in\mathbb{Z}_+$, we can claim that expression (4.1) is a true asymptotical solution to the Lax equation (4.2). □

Recalling now that the expression

$$
\gamma(\lambda):=-\sum_{n=0}^{N-1}\ln h_n+\sum_{n=0}^{N-1}\ln(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})
\tag{4.9}
$$

as $\lambda \to \infty$ is a generating function of conservation laws for the dynamical system (1.2), one finds that functionals

$$
\begin{aligned}
\bar{\gamma}_0 &= \sum_{n=0}^{N-1} \ln(1 - v_n u_n), \gamma_0 = -\sum_{n=0}^{N-1} \sigma_n^{(0)}, \\
\gamma_1 &= -\sum_{n=0}^{N-1} (\sigma_n^{(1)} + \frac{1}{2}\sigma_n^{(0)}\sigma_n^{(0)}), \\
\gamma_2 &= -\sum_{n=0}^{N-1} (\sigma_n^{(2)} + \frac{1}{3}\sigma_n^{(0)}\sigma_n^{(0)}\sigma_n^{(0)} + \sigma_n^{(0)}\sigma_n^{(1)}), ...,
\end{aligned}
\tag{4.10}
$$

and so on, make up an infinite hierarchy of exact conservative quantities for the discrete nonlinear Schrödinger dynamical system (1.2).

Make here some remarks concerning the complete integrability of the discrete nonlinear Schrödinger dynamical system (1.2). First we can easily enough state, making use of the standard asymptotical small parameter approach [6, 13, 7], that the Nöther equation (1.7) on the manifold $M_{(N)}$ possesses [12, 11] the following exact Poissonian operator solution:

$$
\vartheta_n = \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix},
\tag{4.11}
$$

$n \in \mathbb{Z}_N$, subject to which the dynamical the dynamical system (1.2) is Hamiltonian:

$$
d(u, v)^{\intercal}/dt = -\vartheta \text{ grad } H_{\vartheta}[u, v]
\tag{4.12}
$$

on the periodic manifold $M_{(N)}$, where the Hamiltonian function

$$
H_{\vartheta} := \sum_{n=0}^{N-1} \ln h_n^2 - \sum_{n=0}^{N-1} (v_n u_{n-1} + v_{n-1} u_n -) = 2 \ln \bar{\gamma}_0 - \frac{1}{2}(\gamma_0 + \gamma_0^*).
\tag{4.13}
$$

By means of similar, but more cumbersome calculations, one can find the second Poissonian operator solution to the Nöther equation (1.7) in the following matrix form:

$$
\begin{aligned}
\eta_n &= \begin{pmatrix} (h_n - u_n D_n^{-1} u_n)\Delta & (u_n^2 + u_n D_n^{-1} u_n)\Delta^{-1} \\ v_n D_n^{-1} v_n \Delta & -(1 + v_n D_n^{-1} u_n)\Delta^{-1} \end{pmatrix} \times \\
&\times \begin{pmatrix} u_n D_n^{-1} u_n & (h_n - u_n D_n^{-1} v_n \\ 1 + v_n D_n^{-1} u_n & -(v_n + v_n D_n^{-1} v_n) \end{pmatrix},
\end{aligned}
\tag{4.14}
$$

where the operation $D_n^{-1}(...) := \frac{1}{2}[\sum_{k=0}^{n-1}(...)_k - \sum_{k=n}^{N-1}(...)_k]$ is quasi-skew-symmetric with respect to the usual bi-linear form on $T^*(M_{(N)}) \times T(M_{(N)})$, satisfying the operator identity $(D_n^{-1})^* = -\Delta^{-1} D_n^{-1} \Delta$, $n \in \mathbb{Z}$.

The Poissonian operators (4.11) and (4.14) are compatible, that makes it possible to construct by means of the algebraic gradient-holonomic algorithm the related Lax type representation for the dynamical system (1.2). The corresponding result is as follows: the discrete linear spectral problem

$$
\Delta f_n = l_n[u, v; \lambda] f_n,
\tag{4.15}
$$

where $f \in l_\infty(\mathbb{Z}; \mathbb{C}^2)$ and for $n \in \mathbb{Z}$

$$
l_n[u, v; \lambda] = \begin{pmatrix} \lambda & u_n \\ v_n & \lambda^{-1} \end{pmatrix},
\tag{4.16}
$$

allows the linear Lax type isospectral evolution

$$
df_n/dt = p_n(l) f_n
\tag{4.17}
$$

for some matrix $p_n(l) \in End\ \mathbb{C}^2, n \in \mathbb{Z}$, equivalent to the Hamiltonian flow

$$
df_n/dt = \{H_{\vartheta}, f_n\}_{\vartheta},
\tag{4.18}
$$

where $\{.,.\}_\vartheta$ is the corresponding to (4.11) Poissonian structure on the manifold $M_{(N)}$. The equivalence of (4.11) and (4.18) can be easily enough demonstrated, if to construct the corresponding to (4.15) monodromy matrix $S_n(\lambda), n \in \mathbb{Z}$, for all $\lambda \in \mathbb{C}$ and to calculate the Hamiltonian evolution

$$\frac{d}{dt} S_n(\lambda) = \{H_\vartheta, S_n(\lambda)\}_\vartheta = [p_n(l), S_n(\lambda)], \tag{4.19}$$

giving rise to the same matrix $p_n(l) \in End\mathbb{C}^2, n \in \mathbb{Z}$, as that entering equation (4.17).

Thus, we have shown that the nonlinear discrete Schrödinger dynamical system (1.2) is Lax type integrable bi-Hamiltonian flow on the manifold $M_{(N)}$. Since the solution $\varphi(\lambda) \in T^*(M_{(N)})$, constructed above, satisfies the gradient-like relationship

$$\lambda\vartheta\ \varphi(\lambda) = \eta\ \varphi(\lambda) \tag{4.20}$$

for all for $\lambda \in \mathbb{C}$, we derive that the found above conservation laws are commuting to each other with respect to both Poisson brackets $\{.,.\}_\vartheta$ and $\{.,.\}_\vartheta$. The latter gives rise to the classical Liouville integrability [2, 15] of the discrete nonlinear Schrödinger dynamical system (1.2) on the periodic manifold $M_{(N)}$. The detail analysis of the integrability procedure via the mentioned before Bogoyavlensky- Novikov reduction [16, 4] and the explicit construction of solutions to the dynamical system (1.2) are planned to be presented in a separate work.

5. Example 2: the Ragnisco-Tu differential-difference dynamical system and its integrability

Consider the Ragnisco-Tu differential-difference dynamical system (1.3), defined on the periodic manifold $M_{(N)} \subset l_\infty(\mathbb{Z};\mathbb{C}^2)$, and construct first the corresponding asymptotical solution to the Lax equation (1.11). The following lemma holds.

Lemma 5.1. *The functional expression*

$$\varphi_n := \begin{pmatrix} a_n(\lambda) \\ 1 \end{pmatrix} \exp(\lambda t) \prod_{j=1}^{n} \sigma_j(\lambda), \tag{5.1}$$

is an asymptotical, as $\lambda \to \infty$, *solution to the determining Lax equation (1.11) for all* $n \in \mathbb{Z}_N$ *with the operator* $K^{\prime,*} : T^*(M_{(N)}) \to T^*(M_{(N)})$ *of the form:*

$$K_n^{\prime,*} = \begin{pmatrix} \Delta^{-1} - 2u_n v_n & v_n^2 \\ -u_n^2 & -\Delta + 2u_n v_n \end{pmatrix}. \tag{5.2}$$

where, by definition,

$$\begin{aligned} \sigma_n(\lambda) &\simeq \lambda(1 - \sum_{s\in\mathbb{Z}_+} \sigma_n^{(s)}[u,v]\lambda^{-s}), \\ a_n(\lambda) &\simeq \sum_{s\in\mathbb{Z}_+} a_n^{(s)}[u,v]\lambda^{-s}, \end{aligned} \tag{5.3}$$

and the following analytical expressions

$$\begin{aligned} \sigma_n^{(0)} &= 0, a_n^{(0)} = 0; \sigma_n^{(1)} = -2u_{n-1}v_{n-1}, a_n^{(1)} = -v_n^2; \\ \sigma_n^{(2)} &= 2u_{n-1}v_{n-2} - u_{n-1}^2 v_{n-1}^2, a_n^{(2)} = 2v_n(v_{-n-1} - v_n^2 u_n); \\ \sigma_n^{(3)} &= -2u_{n-1}v_{n-2} - D_n^{-1}(d\sigma_n^{(2)}/dt + \sigma_n^{(1)} d\sigma_n^{(1)}/dt), \\ a_n^{(3)} &= -da_n^{(2)}/dt - 2(u_{n-1}v_{n-2}v_n^2 - u_n v_n v_{n-1}^2), ..., \end{aligned} \tag{5.4}$$

and so on, hold.

Proof. It is easy to calculate that local σ- and a-functionals on $M_{(N)}$ satisfy the following functional equations:

$$\begin{gathered} \lambda(1 - \sigma_n(\lambda)) + D_n^{-1} \tfrac{d}{dt} \ln \sigma_n(\lambda) - u_n^2 a_n(\lambda) + 2u_n v_n = 0, \\ da_n(\lambda)/dt + \lambda a_n(\lambda) + a_n(\lambda) D_n^{-1} \tfrac{d}{dt} \ln \sigma_n(\lambda) - 2u_n v_n \lambda^{-1} a_{n-1}(\lambda)\sigma_n(\lambda)^{-1} + v_n^2 = 0, \end{gathered} \tag{5.5}$$

which allow the asymptotical as $\lambda \to \infty$ solutions in the form (5.3). Then, solving step-by-step the corresponding recurrent equations one finds the exact analytical expressions (5.4). Taking now into account that for each $n \in \mathbb{Z}_+$ there exists such a local functional $\rho_n(\lambda)$ that the expression $\frac{d}{dt} \ln \sigma_n(\lambda) = D_n \rho_n(\lambda)$ holds on $M_{(N)}$ identically, we derive that the functional expression (5.1) solves the Lax equation (1.11), proving the lemma. □

As a simple corollary of Lemma 5.1 we obtain that the expression

$$\gamma(\lambda) := \sum_{n=1}^{N} \ln(1 - \sum_{s\in\mathbb{Z}_+} \sigma_n^{(s)} \lambda^{-s-1}) \simeq \sum_{j\in\mathbb{Z}_+} \gamma_j \lambda^{-j} \tag{5.6}$$

is a generating functional for the infinite hierarchy of conservation laws $\gamma_j \in D(M_{(N)}), j \in \mathbb{Z}_+$, of the Ragnisco-Tu differential-difference dynamical system (1.3).

Show now that the Ragnisco-Tu differential-difference dynamical system (1.3) is a bi-Hamiltonian dynamical system on the functional manifold $M_{(N)}$. Really, based on Lemma 1.4, we can find that the element $\psi := \frac{1}{2}(v_n, -u_n)^\intercal \in T^*(M_{(N)})$ satisfies the functional equation (1.16):

$$d\psi/dt + K^{',*}\psi = \text{grad } \mathcal{L}, \ \mathcal{L} = -\frac{1}{2}\sum_{k=0}^{N-1} u_n^2 v_n^2, \tag{5.7}$$

giving rise to the first Poissonian structure

$$\vartheta_n := \psi_n' - \psi_n'^{,*} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.8}$$

on the manifold $M_{(N)}$ with respect to which the differential-difference dynamical system (1.3) is Hamiltonian:

$$\frac{d}{dt}(u_n, v_n)^\intercal = -\vartheta_n \text{ grad } H_{\vartheta,n}[u,v], \tag{5.9}$$

where the Hamiltonian function, owing to relationship (1.22), equals

$$H_\vartheta := (\psi, K) - \mathcal{L}_\psi) = \sum_{k=0}^{N-1}(u_n^2 v_n^2/2 - u_n v_{n-1}) = -\frac{1}{2}\sum_{k=0}^{N-1}\sigma_n^{(2)}. \tag{5.10}$$

The same way one can find the second compatible with (5.8) Poissonian operator

$$\eta_n := \begin{pmatrix} -u_n^2 + 2u_n D_n^{-1}\Delta u_n & \Delta - 2u_n D_n^{-1}\Delta v_n \\ -\Delta^{-1} + 2u_n v_n - 2v_n D_n^{-1}\Delta u_n & -v_n^2 + 2v_n D_n^{-1}\Delta v_n \end{pmatrix}, \tag{5.11}$$

for which

$$\frac{d}{dt}(u_n, v_n)^\intercal = -\eta_n \text{ grad } H_{\eta,n}[u,v], \tag{5.12}$$

where the Hamiltonian function

$$H_\eta := -\sum_{k=1}^{N} u_n v_n = \frac{1}{2}\sum_{k=1}^{N}\sigma_{n+1}^{(1)}. \tag{5.13}$$

Moreover, we claim that the hierarchy of conservation laws (5.6) satisfies as $\lambda \to \infty$ the gradient relationship

$$\lambda\vartheta \text{ grad } \gamma(\lambda) = \eta \text{ grad } \gamma(\lambda), \tag{5.14}$$

entailing their commutation with respect to both Poissonian structures (5.8) and (5.11). The latter allows us to argue that the Ragnisco-Tu differential-difference dynamical system (1.3) is a completely integrable bi-Hamiltonian dynamical system on the manifold $M_{(N)}$.

Since the gradient relationship (5.14) gives rise to the following 'adjoint' Lax type representation

$$d\Lambda/dt = [\Lambda, K^{',*}], \tag{5.15}$$

where, by definition, the expression $\Lambda := \vartheta^{-1}\eta : T^*(M_{(N)}) \to T^*(M_{(N)})$ is called a *recursion* operator. Based on the gradient relationship (5.14) and expression (2.9) we can obtain, within the gradient holonomic approach, that the Ragnisco-Tu differential-difference dynamical system (1.3) is also Lax type integrable whose standard linear shift Lax type spectral problem equals

$$\Delta f_n = l_n[u,v;\lambda] f_n, \quad l_n[u,v;\lambda] = \begin{pmatrix} \lambda + u_n v_n & u_n \\ v_n & 1 \end{pmatrix}, \tag{5.16}$$

for all $n \in \mathbb{Z}, \ \lambda \in \mathbb{C}$, where $(u,v) \in M_{(N)}$ and $f \in l_\infty(\mathbb{Z};\mathbb{C}^2)$.

6. Example 3: the generalized differential-difference Riemann-Burgers and Riemann type dynamical systems and their integrability

In [29, 31, 30] there was analyzed by means of simple enough differential-algebraic tools a generalized (owing to D. Holm and M. Pavlov) Riemann type hydrodynamical hierarchy of equations

$$D_t^s u=0, \quad D_t:=\partial/\partial t+uD_x, D_x:=\partial/\partial x, \tag{6.1}$$

on a smooth functional manifold $\bar{\mathcal{M}}\subset C^\infty(\mathbb{R};\mathbb{R})$ for any integer $s\in\mathbb{Z}_+$ and proved their both bi-Hamiltonian structure and Lax type integrability. For $s=2$ equation (6.1) possesses a Lax type representation, whose Lax l-operator is given [30] by the expression

$$l[u,v;\lambda]=\begin{pmatrix} -u_x\lambda & -v_x \\ 2\lambda^2 & u_x\lambda \end{pmatrix}, \tag{6.2}$$

where the vector-function $(u,v)^\intercal\in\mathcal{M}\subset C^\infty(\mathbb{R};\mathbb{R}^2)$ satisfies the equivalent to (6.1) nonlinear dynamical system

$$D_t u=v, \quad D_t v=0, \tag{6.3}$$

and $\lambda\in\mathbb{C}$ is an arbitrary invariant spectral parameter. For studying differential-difference versions of equations (6.3) we will apply to the Lax l-operator (6.2) the Ablowitz-Ladik discretization scheme [5]. As a result of simple calculations we obtain the following new discrete Lax type spectral problem:

$$f_{n+1}=l_n[u,v;\lambda]f_n, \qquad l_n[u,v;\lambda]=\begin{pmatrix} 1-\lambda D_n u_n & -D_n v_n \\ 2\lambda^2 & 1+\lambda D_n u_n \end{pmatrix} \tag{6.4}$$

for $n\in\mathbb{Z}$, where function $f\in l_\infty(\mathbb{Z};\mathbb{C}^2)$, the vector $(u,v)^\intercal\in M$, if the resulting discrete dynamical system is considered on an N-periodical discrete manifold $M\subset(\mathbb{R}^2)^{\mathbb{Z}_N}$.

To study the related with the spectral problem (6.4) nonlinear differential-difference dynamical systems, we will make use of a slightly generalized gradient-holonomic scheme.

First, we formulate the following simple but useful lemma.

Lemma 6.1. *The following matrix*

$$\tilde{F}_{m,n}(\lambda)=\begin{pmatrix} \tilde{e}^{(1)}_{m,n}(\lambda) & -D_m\hat{u}_m(\lambda)\tilde{e}^{(2)}_{m,n}(\lambda)/(2\lambda) \\ -2\lambda\tilde{e}^{(1)}_{m,n}(\lambda)/D_m\check{u}_m(\lambda) & \tilde{e}^{(2)}_{m,n}(\lambda) \end{pmatrix}, \tag{6.5}$$

where

$$\begin{aligned} \tilde{e}^{(1)}_{m,n}(\lambda) \;&:= \prod_{k=n}^{m-1}[1-\lambda D_k u_k+\lambda(D_k u_k)^2/D_k\check{u}_k(\lambda)], \\ \tilde{e}^{(2)}_{n,m}(\lambda) \;&:= \prod_{k=n}^{m-1}[1+\lambda(D_k u_k-D_k\hat{u}_k(\lambda))], \end{aligned} \tag{6.6}$$

jointly with functional relationships

$$\begin{pmatrix} D_n\check{u}_n(\lambda) \\ D_n\hat{u}_n(\lambda) \end{pmatrix}=\begin{pmatrix} D_n u_n+\lambda^{-1}[1+\frac{D_n\check{u}_n(\lambda)-D_n u_n}{D_n^2\check{u}_n(\lambda)}]^{-1} \\ D_n u_n+\lambda^{-1}[1+\frac{D_n\hat{u}_n(\lambda)-D_n u_n}{D_n^2\hat{u}_n(\lambda)}]^{-1} \end{pmatrix} \tag{6.7}$$

and

$$2D_n v_n-(D_n u_n)^2=0, \tag{6.8}$$

solves the associated with spectral problem (6.4) linear matrix equation

$$\tilde{F}_{m+1,n}(\lambda)=l_m[u,v;\lambda]\tilde{F}_{m,n}(\lambda) \tag{6.9}$$

under the initial condition

$$\tilde{F}_{m,n}(\lambda)|_{m=n}=\mathbf{I}+O(1/\lambda) \tag{6.10}$$

as $\lambda\to\infty$ for all $m,n\in\mathbb{Z}$.

As a corollary from relationships (6.7) one easily finds limits

$$\lim_{\lambda\to\infty} D_n\check{u}_n(\lambda) = D_n u_n, \quad \lim_{\lambda\to\infty} D_n\hat{u}_n(\lambda) = D_n u_n, \tag{6.11}$$

uniformly holding for all $n \in \mathbb{Z}$.

Now, based on the matrix expressions (6.5) and (6.6), one can construct the fundamental matrix

$$F_{m,n}(\lambda) := \tilde{F}_{m,n}(\lambda)\tilde{F}_{n,n}^{-1}(\lambda), \tag{6.12}$$

solving the linear problem

$$F_{m+1,n}(\lambda) = l_m[u,v;\lambda]F_{m,n}(\lambda) \tag{6.13}$$

under the initial condition

$$F_{m,n}(\lambda)|_{m=n} = \mathbf{I} \tag{6.14}$$

for all $n \in \mathbb{Z}$ as $\lambda \to \infty$.

Taking into account that the manifold M is N - periodic, one can construct the next important objects - the asymptotical as $\lambda \to \infty$ monodromy matrix

$$S_n(\lambda) := F_{n+N,n}(\lambda) \tag{6.15}$$

for any $n \in \mathbb{Z}$. By construction, the monodromy matrix (6.15) satisfies the following useful properties:

$$S_{n+N}(\lambda) := S_n(\lambda), \quad \det S_n(\lambda) = 1, \tag{6.16}$$

holding for all $n \in \mathbb{Z}$ and $\lambda \to \infty$.

Keeping now in mind the importance of invariants and Poissonian structures related with the linear spectral problem (6.4), we proceed to studying its basic Lie-algebraic properties and connections with the so called vertex operator representation [32, 33] of the related whole hierarchy of integrable differential-difference dynamical systems on the manifold M.

Namely, we will sketch below the Lie-algebraic aspects [34, 33, 35, 36] of differential-difference dynamical systems, associated with our Lax-type linear difference spectral problem (6.4), where one assumes that the matrix $l_n := l_n[u,v;\lambda] \in G_n := GL_2(\mathbb{C}) \otimes \mathbb{C}(\lambda,\lambda^{-1})$ for $n \in \mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ as $\lambda \to \infty$. To describe the related Lax type integrable dynamical systems, define first the matrix product-group $G^N := \overset{N}{\underset{j=1}{\otimes}} G_j$ and its action $G^N \times M_G^{(N)} \to M_G^{(N)}$ on the phase space $M_G^{(N)} := \{l_n \in G_n : n \in \mathbb{Z}_N\}$, given as

$$\{g_n \in G_n : n \in \mathbb{Z}_N\} \times \{l_n \in G_n : n \in \mathbb{Z}_N\} = \{g_n l_n g_{n+1}^{-1} \in G_n : n \in \mathbb{Z}_N\}. \tag{6.17}$$

Subject to action (6.17) a functional $\gamma \in \mathcal{D}(M_G^{(N)})$ is invariant iff the following discrete relationship

$$\operatorname{grad}\gamma(l_n)l_n = l_{n+1}\operatorname{grad}\gamma(l_{n+1}) \tag{6.18}$$

holds for all $n \in \mathbb{Z}_N$.

Assume further that the matrix group G^N is identified with its tangent spaces $T_l(G^N)$, $l \in G^N$, locally isomorphic to the Lie algebra $\mathcal{G}^{(N)}$, where $\mathcal{G}^{(N)}$ is the corresponding Lie algebra of the Lie group G^N, isomorphic, by definition, to the tangent space $T_e(G^N)$ at the group unity $e \in G^N$. With any element $l \in G^N$ there are associated, respectively, the left $\eta_l : \mathcal{G}^{(N)} \to T_l(G^N)$ and right $\rho_l : \mathcal{G}^{(N)} \to T_l(G^N)$ differentials of the left and right shifts on the Lie group G^N, and their adjoint mappings $\rho_l^* : T_l^*(G^N) \to \mathcal{G}^{(N),*}$ and $\eta_l^* : T_l^*(G^N) \to \mathcal{G}^{(N),*}$, where

$$\begin{aligned} (\rho_l^*\operatorname{grad}\gamma(l), X) &= (\operatorname{grad}\gamma(l), Xl) = (l\ \operatorname{grad}\gamma(l), X) := tr(l\ \operatorname{grad}\gamma(l)X), \\ (\eta_l^*\operatorname{grad}\gamma(l), X) &= (\operatorname{grad}\gamma(l), lX) = (\operatorname{grad}\gamma(l)l, X) := tr(\operatorname{grad}\gamma(l)lX) \end{aligned} \tag{6.19}$$

for any $X \in \mathcal{G}^{(N)}$ and smooth functional $\gamma \in \mathcal{D}(G^N)$, $tr : G^N \to \mathbb{C}$ is a trace-operation on the group $G^N : trA := res_{\lambda=\infty} \sum_{j\in\mathbb{Z}_N} SpA_j[u,v;\lambda]$ for any $A \in G^N$. Owing to (6.18) and (6.19) we can define the set

$$\{\Phi_n = \operatorname{grad}\gamma(l_n)l_n \ \in \mathcal{G}_n^* := T_e^*(G), \quad n \in \mathbb{Z}_N\} \tag{6.20}$$

belonging to the space $\mathcal{G}^{(N),*} \simeq T_e^*(G^N)$ and satisfying the following invariance property:

$$\Phi_{n+1} = Ad_{l_n}^*\Phi_n(\lambda) = l_n^{-1}\Phi_n(\lambda)l_n \tag{6.21}$$

for any $n \in \mathbb{Z}_N$. The relationship (6.21) allows to define a function $\varphi : G^N \to \mathbb{C}$ invariant with respect to the adjoint action

$$G_n \times G_n \ni (g, S_n(\lambda)) \to ad_g S_n(\lambda) = g S_n(\lambda) g^{-1} \in G_n \tag{6.22}$$

for any $n \in \mathbb{Z}_N$ and such that

$$\gamma(l) = \varphi[S_N(\lambda)], \;\; \Phi_N = \mathrm{grad}\varphi[S_N(\lambda)] S_N(\lambda), \tag{6.23}$$

where, by definition, the expression

$$S_N(\lambda) = \overleftarrow{\prod_{j=1}^{N}} l_j[u, v; \lambda] \tag{6.24}$$

coincides exactly with the proper monodromy matrix for the linear spectral problem (6.4). Since, owing to (6.21), the matrices $\Phi_n = \mathrm{grad}\varphi[S_n(\lambda)] S_n(\lambda) \in \mathcal{G}_n^*$, $n \in \mathbb{Z}_N$, can be reconstructed from (6.24), we find [34, 36] the following Poissonian flow on the matrices $S_n(\lambda) \in G_n, n \in \mathbb{Z}_N$:

$$dS_n(\lambda)/dt = [\mathcal{R}(\mathrm{grad}\varphi[S_n(\lambda)] S_n(\lambda)), S_n(\lambda)] \tag{6.25}$$

with respect to the invariant Casimir function $\varphi \in I(\mathcal{G}_n^*)$ and the quadratic Poissonian structure

$$\{\gamma_1, \gamma_2\} := (l, [\mathrm{grad}\gamma_1(l), \mathcal{R}(l\ \mathrm{grad}\gamma_2(l))] + [\mathcal{R}(l\ \mathrm{grad}\gamma_1(l)), \mathrm{grad}\gamma_2(l)]) \tag{6.26}$$

for any functionals $\gamma_1, \gamma_2 \in \mathcal{D}(G^N)$, constructed by means of a skew-symmetric $\mathcal{R}$-structure $\mathcal{R} : \mathcal{G}^{(N),*} \to \mathcal{G}^{(N)}$. In particular, the equality

$$[\mathrm{grad}\varphi(S_n), S_n] = 0 \tag{6.27}$$

holds for all $n \in \mathbb{Z}_N$.

Having taken into account (6.23), one can rewrite (6.25) in the following equivalent form:

$$dS_n/dt = [\mathcal{R}(\mathrm{grad}\gamma(l_n) l_n), S_n], \tag{6.28}$$

holding for all $n \in \mathbb{Z}_N$. The latter jointly with (6.21) makes it possible to retrieve [32, 35] the related evolution of elements $l_n \in G_n$, $n \in \mathbb{Z}_N$:

$$\begin{aligned} dl_n/dt &= p_{n+1}(l) l_n - l_n p_n(l), \\ p_n(l) &: = \mathcal{R}(\mathrm{grad}\gamma(l_n) l_n) \end{aligned} \tag{6.29}$$

and following from the relationships

$$\begin{aligned} S_n(\lambda) &= \psi_n(l) S_N(\lambda) \psi_n^{-1}(l), \\ \psi_n(l) &= \overleftarrow{\prod_{j=1}^{n}} l_j[u, v; \lambda]. \end{aligned} \tag{6.30}$$

Subject to the linear spectral problem (6.4) the solution $f \in l_\infty(\mathbb{Z}, \mathbb{C}^2)$ satisfies the associated temporal evolution equation

$$df_n/dt = p_n(l) f_n \tag{6.31}$$

for any $n \in \mathbb{Z}$. It is easy to check that the compatibility condition of linear equations (6.4) and (6.31) is equivalent to the discrete Lax type representation (6.29), which, upon reducing it on the group manifold M_G, gives rise to the corresponding nonlinear Lax type integrable dynamical system on the discrete manifold M. It follows from the fact that all Casimir invariant functions, when reduced on the manifold M_G, are in involution [35, 36] with respect to the Poisson bracket (6.26).

In the case when the skew-symmetric $\mathcal{R}$-structure $\mathcal{R} = 1/2(P_+ - P_-)$, where $P_\pm : \mathcal{G}_n \to \mathcal{G}_{n,\pm} \subset \mathcal{G}_n$ are the projectors on the λ-positive and λ- negative, respectively, degree subalgebras of the Lie algebra $\mathcal{G}_n$, the determining Lax type equation (6.29) generates the flows

$$\frac{d}{dt_j} l_n[u, v; \lambda] = (\lambda^{j+1} \tilde{S}_{n+1}(\lambda))_+ l_n[u, v; \lambda] - l_n[u, v; \lambda](\lambda^{j+1} \tilde{S}_n(\lambda))_+ \tag{6.32}$$

for all $j \in \mathbb{Z}_+$, where $\tilde{S}_n(\lambda)$, $n \in \mathbb{Z}_N$, are the corresponding asymptotical expansion of the suitably normalized monodromy matrix $\tilde{S}_n(\lambda) \in \mathcal{G}_-, n \in \mathbb{Z}_N$, as $\lambda \to \infty$.

The hierarchies of evolution equations (6.32) can be rewritten as the following generating flows:

$$\frac{d}{dt_{(\mu)}} l_n[u, v; \lambda] = \frac{\lambda\mu}{\mu - \lambda} [\tilde{S}_{n+1}(\mu) l_n[u, v; \lambda] - l_n[u, v; \lambda] \tilde{S}_n(\mu)] \tag{6.33}$$

as $\lambda \to \infty$ and $|\lambda/\mu| < 1$, where, by definition,

$$\frac{d}{dt_{(\mu)}} = \sum_{j\in\mathbb{Z}_+} \mu^{-j} \frac{d}{dt_j}. \tag{6.34}$$

Proceed now to describing the analytical structure of the regularized matrices $\tilde{S}_n(\mu)$, $n \in \mathbb{Z}_N$, for arbitrary $\mu \in \mathbb{C}$. To do this effectively we need to consider the corresponding to (6.33) evolution equations for the monodromy matrix $S_n(\lambda)$ as $\lambda \to \infty$, which make it possible to construct the related flows on functions $\breve{a}_n := D_n\breve{u}_n(\lambda)$ and $\hat{a}_n := D_n\hat{u}_n(\lambda), n \in \mathbb{Z}_N$, represented [32, 33] by means of the related vertex operators action $\hat{X}_\lambda : \bar{M}^2 \to \bar{M}^2$, where $\bar{M} := \mathbb{R}^{\mathbb{Z}_N}$,

$$\begin{aligned} \hat{X}_\lambda \quad &: \quad = (\exp D_\lambda, \exp(-D_\lambda))^\intercal, \\ D_\lambda \quad &: \quad = \sum_{j\in\mathbb{Z}_+} \frac{1}{(j\lambda^j)} \frac{d}{dt_j}, \end{aligned} \tag{6.35}$$

and

$$\hat{X}_\lambda \left(\begin{array}{c} u \\ u \end{array} \right) = \left(\begin{array}{c} u(t_0 + 1/\lambda, t_1 + 1/(2\lambda^2), ...) \\ u(t_0 - 1/\lambda, t_1 - 1/(2\lambda^2), ...) \end{array} \right), \tag{6.36}$$

as $\lambda \to \infty$.

Now, based on the flows 6.33, one can derive the corresponding evolution equation on the monodromy matrix $\tilde{S}_n(\lambda) \in \mathcal{G}_-, n \in \mathbb{Z}_N$, with respect to the vector field (6.34):

$$\frac{d}{dt_{(\mu)}} \tilde{S}_n(\lambda) = \frac{\lambda\mu}{\mu - \lambda} [(\tilde{S}_n(\mu), \tilde{S}_n(\lambda)] \tag{6.37}$$

as $|\lambda/\mu| < 1, \lambda \to \infty$, which entails upon taking the limit $\mu \to \lambda$ the equation

$$\frac{d}{dt} \tilde{S}_n(\lambda) = \lambda^2 [\frac{d}{d\lambda} \tilde{S}_n(\lambda), \tilde{S}_n(\lambda)], \tag{6.38}$$

where we put, by definition,

$$\frac{d}{dt} := \sum_{j\in\mathbb{Z}_+} \lambda^{-j} \frac{d}{dt_j}. \tag{6.39}$$

Now, taking into account Lemma 6.1, the related matrix expressions (6.5), (6.12) and (6.15) and having analyzed the analytical structure of the resulting monodromy $\tilde{S}_n(\lambda) \in \mathcal{G}_-, n \in \mathbb{Z}_N$, one can state by means of simple but slightly cumbersome calculations the following proposition.

Proposition 6.2. *The following differential relationships*

$$\left(\begin{array}{c} \frac{d}{dt} \left[\frac{2\lambda s_n^{(11)}}{s_n^{(21)}} \left(1 - \sqrt{1 - \frac{s_n^{(12)} s_n^{(21)}}{s_n^{(11)} s_n^{(11)}}} \right) \right] = -\lambda^2 \frac{d}{d\lambda} \left[\frac{2\lambda s_n^{(11)}}{s_n^{(21)}} \left(1 - \sqrt{1 - \frac{s_n^{(12)} s_n^{(21)}}{s_n^{(11)} s_n^{(11)}}} \right) \right], \\ \frac{d}{dt} \left[-\frac{2\lambda s_n^{(11)}}{s_n^{(21)}} \left(1 + \sqrt{1 - \frac{s_n^{(12)} s_n^{(21)}}{s_n^{(11)} s_n^{(11)}}} \right) \right] = \lambda^2 \frac{d}{d\lambda} \left[-\frac{2\lambda s_n^{(11)}}{s_n^{(21)}} \left(1 + \sqrt{1 - \frac{s_n^{(12)} s_n^{(21)}}{s_n^{(11)} s_n^{(11)}}} \right) \right], \end{array} \right), \tag{6.40}$$

jointly with equalities

$$\begin{aligned} D_n\hat{u}_n(\lambda) &= \frac{2\lambda s_n^{(11)}}{s_n^{(21)}} \left(1 - \sqrt{1 - \frac{s_n^{(12)} s_n^{(21)}}{s_n^{(11)} s_n^{(11)}}} \right), \\ D_n\breve{u}_n(\lambda) &= -\frac{2\lambda s_n^{(11)}}{s_n^{(21)}} \left(1 + \sqrt{1 - \frac{s_n^{(12)} s_n^{(21)}}{s_n^{(11)} s_n^{(11)}}} \right), \end{aligned} \tag{6.41}$$

hold as $\lambda \to \infty$ for all $n \in \mathbb{Z}_N$.

As a corollary of (6.11) and differential relationships (6.40) one obtains easily the following vertex operator representation (6.36) for the functions (6.41):

$$\left(\begin{array}{c} D_n\hat{u}_n(\lambda) \\ D_n\breve{u}_n(\lambda) \end{array} \right) = \left(\begin{array}{c} D_n u_n(t_0 + 1/\lambda, t_1 + 1/(2\lambda^2), ...) \\ D_n u_n(t_0 - 1/\lambda, t_1 - 1/(2\lambda^2), ...) \end{array} \right) = \hat{X}_\lambda \left(\begin{array}{c} D_n u_n \\ D_n u_n \end{array} \right), \tag{6.42}$$

holding as $\lambda \to \infty$ for all $n \in \mathbb{Z}_N$. Recalling additionally the algebraic relationships (6.7) and (6.8), one finds from (6.42) the related infinite hierarchy of differential-difference dynamical systems on the manifold M :

$$\frac{d}{dt_j}\left(\begin{array}{c} D_n u_n \\ D_n v_n \end{array}\right) = \left(\begin{array}{c} c_j (D_n^2 u_n)^{-j} \\ 2c_j v_n (D_n u_n)^{-1} (D_n^2 u_n)^{-j} \end{array}\right) \tag{6.43}$$

for all $j \in \mathbb{Z}_+$, where $c_j \in \mathbb{R}, j \in \mathbb{Z}_+$, are some recurrently calculated constant coefficients.

Thus, we see that the constructed hierarchy (6.43) does not contain a "naive" discretization of the generalized Riemann type system of equations (6.3) in spite of the fact that the difference Lax type linear spectral problem (6.4) regularly reduces to the corresponding linear spectral problem for dynamical system (6.3).

Nonetheless, such a situation is not faced in the case of the inviscid discrete Riemann-Burgers dynamical system (1.4):

$$dw_n/dt = w_n(w_{n+1} - w_{n-1})/2 := K_n[w], \tag{6.44}$$

defined on an N-periodical discrete manifold $M \subset l_2(\mathbb{Z}_N; \mathbb{R})$. Following the gradient-holonomic scheme of studying the integrability of (6.44), we first state the existence of an infinite hierarchy of conservation laws and the corresponding bi-Hamiltonian formulation.

Consider, owing to Proposition , the determining equation (1.11)

$$d\varphi_n/dt - [(\Delta - \Delta^{-1})w_n/2 - (w_{n+1} - w_{n-1})/2]\varphi_n = 0 \tag{6.45}$$

and its asymptotical solution $\varphi \in T^*(M)$ in the form (2.13):

$$\varphi_n = \prod_{j=0}^{n-1} \sigma_j[w; \lambda], \tag{6.46}$$

where $n \in \mathbb{Z}$ and the local functionals $\sigma_j[w;\lambda], j \in \mathbb{Z}_+$, possess as $\lambda \to \infty$ the following expansions

$$\sigma_j[w;\lambda] \simeq \sum_{s \in \mathbb{Z}_+} \sigma_j^{(s)}[w]\lambda^{-s}. \tag{6.47}$$

Having solved recurrently the resulting functional equations

$$D_n^{-1}(\ln \sigma_n)_t + (w_{n-1}/\sigma_{n-1} - w_{n+1}\sigma_n)/2 + (w_{n+1} - w_{n-1}) = 0, \tag{6.48}$$

one finds easily the infinite hierarchy (5.6) of conservations laws:

$$\gamma_0 = \sum_{n=0}^{N-1}(w_n + w_{n-1}), \gamma_1 = 0, \gamma_2 = \sum_{n=0}^{N-1}[(w_n + w_{n-1})^2 + w_n(w_{n-1} + w_{n+1})], ..., \gamma_{2j+1} = 0 \tag{6.49}$$

for all $j \in \mathbb{Z}_+$. Now, applying to the hierarchy of conservation laws the approach of Lemma 1.4, one can find by means of slightly cumbersome and lengthy calculations the following pair $\vartheta, \eta : T^*(M) \to T(M)$ of compatible Poissonian operators on the manifold M :

$$\vartheta_n := w_n(\Delta - \Delta^{-1})w_n, \quad \eta_n := (w_n w_{n+1}\Delta^2 - w_n w_{n-1}\Delta^{-2})(w_n + w_{n-1}\Delta^{-1}). \tag{6.50}$$

In particular, there easily obtains the Hamiltonian representation of the Burgers-Riemann system (6.44):

$$dw_n/dt = -\vartheta_n \mathrm{grad}\ H_\vartheta, \ H_\vartheta := -\sum_{n=0}^{N-1}(w_n + w_{n-1})/2. \tag{6.51}$$

Moreover, the first Poissonian structure of (6.50) allows the continuous limit $\lim\limits_{\substack{\Delta x \to 0 \\ n \to \infty}} w_n := w(x)$, if $n\Delta x := x \in \mathbb{R}$, to the well known [20] correct form

$$\vartheta := (w\partial + \partial w)(w + \partial^{-1} w\partial)/2. \tag{6.52}$$

Making use the Poissonian pair (6.50) one can retrieve within the gradient holonomic scheme a Lax type representation related to the inviscid discrete Riemann-Burgers dynamical system (6.44), whose l-operator is given by the following matrix expression:

$$l_n[w;\lambda] = \left(\begin{array}{cc} \lambda & -w_n \\ 1 & 0 \end{array}\right) \tag{6.53}$$

for $n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$. Mention only that the higher flows, generated by the inviscid Burgers-Riemann dynamical system (6.44), have nothing to do with the generalized Riemann type hydrodynamic systems (6.1) and their discrete approximations. Thereby, it is necessary to develop a different approach to constructing their proper integrable discrete Lax type representation, compatible with the related continuous limit.

7. Conclusion

The gradient-holonomic scheme of direct studying Lax type integrability of differential-difference nonlinear dynamical systems described in this work appears to be effective enough for applications in the one-dimensional case, similar to the case [6, 15, 13, 7, 12, 11] of nonlinear dynamical systems defined on spatially one-dimensional functional manifolds. This algorithm makes it possible to construct simply enough an infinite hierarchy of conservation laws as well as to calculate their compatible co-symplectic structures. As it was also shown, the reduced via the Bogoyavlensky-Novikov approach integrable Hamiltonian dynamical systems on the corresponding invariant periodic submanifolds generate finite dimensional Liouville integrable Hamiltonian systems with respect to the canonical Gelfand-Dikiy type symplectic structures. As interesting examples the complete integrability analysis of the nonlinear discrete Schrödinger, the Ragnisco-Tu and Burgers-Riemann type dynamical systems was presented.

Subject to different not-direct approaches to studying the integrability of differential-difference dynamical systems on discrete manifolds it is worth to mention the works [19, 20, 21, 22, 23, 28, 38, 24, 39, 40] based on the inverse spectral transform and related Lie-algebraic methods [25, 27, 38, 24, 39, 40], where there are constructed *a priori* both conservation laws and the corresponding Lax type representations. Concerning these approaches, many of their important analytical properties were constructively absorbed by the gradient-holonomic scheme of this work and realized directly as an algorithm.

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References

[1] Abraham R., Marsden J. Foundation of mechanics. *The Benjamin/Cummings Publ.* Co, Masachusets, 1978
[2] Arnold V.I. Mathematical methods of classical mechanics. NY, *Springer*, 1978
[3] Lax P.D. Periodic solutions of the KdV equation // Commun. Pure and Appl. Math., 1975, 28, N421, p. 141-188
[4] Novikov S.P. (Editor) Theory of Solitons. Moscow, "Nauka", 1980 (in Russian)
[5] Ablowitz M., Ladik J. Nonlinear differenmtial-difference equations. J. Math. Phys.,1975, 16, N3, p. 598-603
[6] Prykarpatsky A. and Mykytyuk I. Algebraic integrability of nonlinear dynamical systems onmanifolds: classical and quantum aspects. *Kluwer Academic Publishers*, the Netherlands, 1998
[7] Hentosh O., Prytula M. and Prykarpatsky A. Differential-geometric and Lie-algebraic foundations of investigating nonlinear dynamical systems on functional manifolds, 2nd edn. Lviv: Lviv University Publication, 2006 (in Ukrainian)
[8] Blaszak M. Multi-Hamiltonian theory of dynamical systems. Berlin-Heidelberg: Springer, 1998
[9] Dodd R.K., Eilbeck J.C., Gibbon J. and Morris H.C. Solitons and nonlinear equations. Academic Press, London, 1984
[10] Magri F. A geometric approach to the nonlinear solvable equations. Lect. Notes Phys., 1980, N120, p. 233-263
[11] Mitropolsky Yu.A. and Samoylenko V.H. Differential-difference dynamical systems, associtated with the difference dirac operator, and their complete integrability. Ukr. Math. J., 1985, 37, N2, p. 180-186 (in Russian)
[12] Prykarpatsky A.K. Elements of the integrability theory of discrete dynamical systems. Ukr. Math. J., 39, N1, p. 298-301 (in Russian)
[13] Blackmore D., Prykarpatsky A.K. and Samoylenko V. Nonlinear dynamical systems of mathematical physics: spectral and differential-geometric integrability analysis. NY, World Scientific, 2010

[14] Bogolubov N.N. (Jr.) and Prykarpatsky A.K. Inverse periodic problem for a discrete approximation of the Nonlinear Schrödinger equation. Dokl. AN SSSR, 1982, 265, N5, p. 1103-1108
[15] Mitropolsky Yu.A., Bogolubov N.N. (Jr.), Prykarpatsky A.K. and Samoylenko V.H. Integrable dynamical systems: differential-geometric and spectral aspects. K.: "Naukova Dumka", 1987 (in Russian)
[16] Bogoyavlensky O.I. and Novikov S.P. On the connection of Hamiltonian formulations of stationary and non-stationary problems. Funk. Anal. Appl., 1976, 10, N1, p. 9-13 (in Russian)
[17] Gelfand I.M. and Dikiy L.A. Integrable nonlinear equations and the Liouville theorem. Funk. Anal. Appl., 1979, 13, N1, p. 8-20 (in Russian)
[18] Fuchsteiner B. and Fokas A. Symplectic structures, Backlund transformatrions and hereditary symmetries. Physica D, 1981, 4, N1, p. 47-66
[19] Kupershmidt B.A. Discrete Lax equations and differential-difference calculus. // Asterisque, 1985, vol. 123, p. 5-212
[20] Kupershmidt B.A. KP or mKP: Noncommutative mathematics of Lagrangian, Hamiltonian, and Integrable systems. American Mathematical Society, v. 78, 2000
[21] Deift P. Li L.-C., Tomei C. Loop groups, discrete versions of some classical integrable systems and rank-2 extensions. Memoirs of the AMS, 1992, vol. 100, No 479, p. 1-101
[22] Moser J., Veselov A. P. Discrete versions of some classical integrable systems and factorization of matrix polynomials. Zurich, ETH, Preprint, 1989, 76p.
[23] Baez J. C., Gillam J. W. An algebraic approach to discrete mechanics. Lett. Math. Phys., 1994, vol. 31, No 3, p. 205-212.
[24] Mikhailov A.V. Canonical conservation laws and integrability conditions for difference equations. (Joint work with Jing Ping Wang and P. Xenitidis) Report for SIDE-9, Varna, Bulgaria, 2010
[25] Levi D., Winternitz P. and Yamilov R.I. Lie point symmetries of differential-difference equations. arXiv:1004.5311v1 [math-ph] 29 Apr 2010
[26] Ragnisco O. and Tu G.Z. A new hierarchy of integrable discrete systems. Preprint of Dept. of Physics, University of Roma, 1989
[27] Pickering A. and Zhu Z.-N. Integrable lattice hierarchies associated with two new (2 +1)-dimensional discrete spectral problems. Physics Letters A 373 (2009) 3944–3951
[28] Blaszak M., Szum A. and Prykarpatsky A. Central extension approach to integrable field and lattice-field systems in (2+1)-dimensions. Reports on Mathematical Physics, 1999.-37, N5, p. 27-32
[29] Prykarpatsky A.K., Artemovych O.D., Popowicz Z. and Pavlov M.V. Differential-algebraic integrability analysis of the generalized Riemann type and Korteweg–de Vries hydrodynamical equations. J. Phys. A: Math. Theor. 43 (2010) 295205 (13pp)
[30] Golenia J., Pavlov M., Popowicz Z. and Prykarpatsky A. On a nonlocal Ostrovsky-Whitham type dynamical system, its Riemann type inhomogenious regularizations and their integrability. SIGMA, 2010, 6, p. 1-13
[31] Golenia J., Bogolubov N. (jr.), Popowicz Z., Pavlov M. and Prykarpatsky A. A new Riemann type hydrodynamical hierarchy and its integrability analysis. *Preprint ICTP,* IC/2009/095, 2009
[32] Jimbo M. and Miwa T. Solutions and infinite-dimensional Lie algebras. Public. RIMS, 19, 1983, p. 943-10001
[33] Newell A.C. Solitons in mathematics and physics. Arizona, SIAM Publ., 1985
[34] Faddeev L.D., Takhtadjan L.A. Hamiltonian methods in the theory of solitons. NY, Springer, 1987
[35] Reyman A.G., Semenov-Tian-Shansky. Integrable systems. Moscow-Izhevsk, R&C-Dynamics, 2003 (in Russian)
[36] Reyman A.G., Semenov-Tian-Shansky.What is the classical R-matyrix? Functional Analysis and Appl., v.17,N4, p. 17-33 (in Russian)
[37] Pavlov M. The Gurevich-Zybin system. J. Phys. A: Math. Gen. 2005, 38, p. 3823-3840
[38] Hentosh O., Prykarpatsky A. and Samoylenko V.H. The Lie-Algebraic structure of Lax type integrable nonlocal differential-difference equations. Journal of Nonlinear Oscillations. 2000, v.3, N1, p.84-94
[39] Hardouin C. and Singer M.F. Differential Galois theory of linear difference equations. Math. Ann., 2008, 342, p. 333–377
[40] Wilson G. On the quasi-Hamiltonian formalism of the KdV equation. Phys. Lett. 132, p. 445–50

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